

ON EVOLUTION EQUATIONS GOVERNED BY NON-AUTONOMOUS FORMS

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ABSTRACT. We consider a linear non-autonomous evolutionary Cauchy problem

$$(0.1) \quad \dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \text{ for a.e. } t \in [0, T], \quad u(0) = u_0,$$

where the operator $\mathcal{A}(t)$ arises from a time depending sesquilinear form $\mathfrak{a}(t, \cdot, \cdot)$ on a Hilbert space H with constant domain V . Recently a result on L^2 -maximal regularity in H , i.e., for each given $f \in L^2(0, T, H)$ and $u_0 \in V$ the problem (0.1) has a unique solution $u \in L^2(0, T, V) \cap H^1(0, T, H)$, is proved in [10] under the assumption that \mathfrak{a} is symmetric and of bounded variation. The aim of this paper is to prove that the solutions of an approximate non-autonomous Cauchy problem in which \mathfrak{a} is symmetric and piecewise affine are closed to the solutions of that governed by symmetric and of bounded variation form. In particular, this provide an alternative proof of the result in [10] on L^2 -maximal regularity in H .

1. INTRODUCTION

In this work we are interested by evolutionary linear equations of the form

$$(1.1) \quad \dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0,$$

where the operators $\mathcal{A}(t)$, $t \in [0, T]$ arise from time dependent sesquilinear forms. More precisely, let H and V denote two separable Hilbert spaces such that V is continuously and densely embedded into H (we write $V \hookrightarrow_d H$). Let V' be the antidual of V and denote by $\langle \cdot, \cdot \rangle$ the duality between V' and V . As usual, we identify H with H' and we obtain that $V \hookrightarrow_d H \cong H' \hookrightarrow_d V'$. These embeddings are continuous and dense (see e.g., [9]). Let

$$\mathfrak{a} : [0, T] \times V \times V \rightarrow \mathbb{C}$$

be a *closed non-autonomous form*, i.e., $\mathfrak{a}(t, \cdot, \cdot)$ is sesquilinear for all $t \in [0, T]$, $\mathfrak{a}(\cdot, u, v)$ is measurable for all $u, v \in V$,

$$|\mathfrak{a}(t, u, v)| \leq M \|u\|_V \|v\|_V \quad (t \in [0, T], u, v \in V)$$

and

$$\operatorname{Re} \mathfrak{a}(t, u, u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad (t \in [0, T], u \in V)$$

for some $\alpha > 0$, $M > 0$ and $\omega \in \mathbb{R}$. The operator $\mathcal{A}(t) \in \mathcal{L}(V, V')$ associated with $\mathfrak{a}(t, \cdot, \cdot)$ on V' is defined for each $t \in [0, T]$ by

$$\langle \mathcal{A}(t)u, v \rangle = \mathfrak{a}(t, u, v) \quad (u, v \in V).$$

Seen as an unbounded operator on V' with domain $D(\mathcal{A}(t)) = V$, the operator $-\mathcal{A}(t)$ generates a holomorphic C_0 -semigroup \mathcal{T}_t on V' . The semigroup is bounded

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on a sector if $\omega = 0$, in which case \mathcal{A} is an isomorphism. We denote by $A(t)$ the part of $\mathcal{A}(t)$ on H ; i.e.,

$$\begin{aligned} D(A(t)) &:= \{u \in V : \mathcal{A}(t)u \in H\} \\ A(t)u &= \mathcal{A}(t)u. \end{aligned}$$

It is a known fact that $-A(t)$ generates a holomorphic C_0 -semigroup T on H and $T = \mathcal{T}|_H$ is the restriction of the semigroup generated by $-\mathcal{A}$ to H . Then $A(t)$ is the operator induced by $\mathbf{a}(t, \cdot, \cdot)$ on H . We refer to [1],[16] and [23, Chap. 2].

In 1961 J. L. Lions proved that the non-autonomous Cauchy problem

$$(1.2) \quad \dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0.$$

has L^2 -maximal regularity in V' :

Theorem 1.1. (*Lions*) For all $f \in L^2(0, T; V')$ and $u_0 \in H$, the problem (1.2) has a unique solution $u \in MR(V, V') := L^2(0, T; V) \cap H^1(0, T; V')$.

Lions proved this result in [18] (see also [24, Chapter 3]) using a representation theorem of linear functionals due to himself and usually known in the literature as *Lions's representation Theorem* and using Galerkin's method in [12, XVIII Chapter 3, p. 620]. We refer also to an alternative proof given by Tanabe [23, Section 5.5].

In Theorem 1.1 only measurability of $\mathbf{a} : [0, T] \times V \times V \rightarrow \mathbb{C}$ with respect to the time variable is required to have a solution $u \in MR(V, V')$. Nevertheless, in applications to boundary valued problems, like heat equations with non-autonomous Robin-boundary-conditions or Schrödinger equations with time-dependent potentials, this is not sufficient. One is more interested in L^2 -maximal regularity in H rather than in V' , i.e., in solutions which belong to

$$(1.3) \quad MR(V, H) := L^2(0, T; V) \cap H^1(0, T; H)$$

rather than in $MR(V, V')$. Lions asked a long time before in [18, p. 68] whether the solution u of (1.2) belongs to $MR(V, H)$ in the case where $\mathbf{a}(t; u, v) = \bar{\mathbf{a}}(t; u, v)$ and $t \mapsto \mathbf{a}(t; u, v)$ is only measurable.

Dier [10] has recently showed that in general the unique assumption of measurability is not sufficient to have $u \in MR(V, H)$. However, several progress are already has been done by Lions [18, p. 68, p. 94,], [18, Theorem 1.1, p. 129] and [18, Theorem 5.1, p. 138] and also by Bardos [8] under additional regularity assumptions on the form \mathbf{a} , the initial value u_0 and the inhomogeneity f . More recently, this problem has been studied with some progress and different approaches by Arendt, Dier, Laasri and Ouhabaz [5], Arendt and Monniaux [6], Ouhabaz [20], Dier [11], Haak and Ouhabaz [19], Ouhabaz and Spina [21]. Results on multiplicative perturbations are also established in [5, 11, 7].

In [15] we proved Theorem 1.1 by a completely different approach developed in [14] and [17]. The method uses an appropriate approximation of the $\mathcal{A}(\cdot)$. Namely, let $\Lambda := (0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n+1} = T)$ be a subdivision of $[0, T]$. Consider the following approximation $\mathcal{A}_\Lambda^S : [0, T] \rightarrow \mathcal{L}(V, V')$ of \mathcal{A} given by

$$\mathcal{A}_\Lambda^S(t) := \begin{cases} \mathcal{A}_k & \text{for } \lambda_k \leq t < \lambda_{k+1}, \\ \mathcal{A}_n & \text{for } t = T, \end{cases}$$

with

$$\mathcal{A}_k u := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} \mathcal{A}(r)u dr \quad (u \in V, k = 0, 1, \dots, n).$$

("S" stands for step). The integral above makes sense since $t \mapsto \mathcal{A}(t)u$ is Bochner integrable on $[0, T]$ with values in V' for all $u \in V$. Note that $\|\mathcal{A}(t)u\|_{V'} \leq M\|u\|_V$

for all $u \in V$ and all $t \in [0, T]$. It is worth to mention that the mapping $t \mapsto \mathcal{A}(t)$ is strongly measurable by the Dunford-Pettis Theorem [2] since the spaces are assumed to be separable and $t \mapsto \mathcal{A}(t)$ is weakly measurable.

It has been proved in [15, Theorem 3.2] that for all $u_0 \in H$ and $f \in L^2(0, T; V')$, the non-autonomous problem

$$(1.4) \quad \dot{u}_\Lambda(t) + \mathcal{A}_\Lambda(t)u_\Lambda(t) = f(t), \quad u_\Lambda(0) = u_0$$

has an (explicit) unique solution $u_\Lambda \in MR(V, V')$, and (u_Λ) converges weakly in $MR(V, V')$ as $|\Lambda| \rightarrow 0$ to the unique solution u of (1.2). If we consider $u_0 \in V$ and $f \in L^2(0, T; H)$ then the solution u_Λ of (1.4) belongs to $MR(V, H) \cap C([0, T]; V)$ (see [17], [15]). If moreover, \mathbf{a} is assumed to be piecewise Lipschitz-continuous on $[0, T]$ then we obtain the convergence of u_Λ in $MR(V, H)$ [15] (see also [5]).

In this paper we are concerned with the recent result obtained in [11]. Instead of functions that are constant on each subinterval $[\lambda_k, \lambda_{k+1}]$, we will consider here those that are linear in time.

2. PRELIMINARY

Let X be a Banach space and $T > 0$. Recall that a point $t \in [0, T]$ is said to be a *Lebesgue point* of a function $f : [0, T] \rightarrow X$ if

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|_X ds = 0.$$

Clearly each point of continuity of f is a Lebesgue point. By [2, Proposition 1.2.2] if f is Bochner integrable then almost all point are Lebesgue points.

Let D be an other Banach space such that D is continuously and densely embedded into X and let $A : [0, T] \rightarrow \mathcal{L}(D, X)$ be a bounded and strongly measurable function, i.e., for each $x \in D$ the function $A(\cdot)x : [0, T] \rightarrow X$ is measurable and bounded.

Let $\Lambda := (0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n+1} = T)$ be a subdivision of $[0, T]$. We consider the following approximations of $A : [0, T] \rightarrow \mathcal{L}(D, X)$ by step operator function $A_\Lambda^S : [0, T] \rightarrow \mathcal{L}(D, X)$ and piecewise linear operator function $A_\Lambda^L : [0, T] \rightarrow \mathcal{L}(D, X)$ given by

$$A_\Lambda^S(t) := \begin{cases} A_k & \text{for } \lambda_k \leq t < \lambda_{k+1}, \\ A_n & \text{for } t = T, \end{cases}$$

and

$$(2.1) \quad A_\Lambda^L(t) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} A_k + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} A_{k+1}, \quad \text{for } t \in [\lambda_k, \lambda_{k+1}],$$

where

$$A_k x := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r)x dr \quad (x \in D, k = 0, 1, \dots, n).$$

Let $|\Lambda| := \max_{j=0,1,\dots,n} (\lambda_{j+1} - \lambda_j)$ denote the mesh of the subdivision Λ . Assume that the subdivision Λ is uniform, i.e., $\lambda_{k+1} - \lambda_k = T/n = |\Lambda|$ for all $k = 0, 1, \dots, n$. In the following Lemma, we show that A_Λ^S and A_Λ^L converge strongly and almost everywhere to A as $|\Lambda| \rightarrow 0$, from which the strong convergence with respect to L^p -norm ($p \in [1, \infty)$) follows.

Lemma 2.1. *Let $A_\Lambda^S : [0, T] \rightarrow \mathcal{L}(D, X)$ be given as above. Then:*

- i) *For all $x \in D$ we have $A_\Lambda^S(t)x \rightarrow A(t)x$ t -a.e. on $[0, T]$ as $|\Lambda| \rightarrow 0$.*
- ii) *$A_\Lambda^S(\cdot)u_\Lambda(\cdot) \rightarrow A(\cdot)u(\cdot)$ in $L^p(0, T; X)$ as $|\Lambda| \rightarrow 0$ if $u_\Lambda \in L^p(0, T; D)$ such that $u_\Lambda \rightarrow u$ in $L^p(0, T; D)$.*

Proof. Let $C \geq 0$ such that $\|A(t)x\|_X \leq C\|x\|_D$ for all $x \in D$ and for almost every $t \in [0, T]$. We have $\|A_k x\|_X \leq C\|x\|_D$ for all $x \in D$ and $k = 0, 1, \dots, n$. Let t be any Lebesgue point of $A(\cdot)x$. Let $k \in \{0, 1, \dots, n\}$ such that $t \in [\lambda_k, \lambda_{k+1})$. Then

$$\begin{aligned} A_\Lambda^S(t)x - A(t)x &= \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} (A(r)x - A(t)x) dr \\ &= \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^t (A(r)x - A(t)x) dr + \frac{1}{\lambda_{k+1} - \lambda_k} \int_t^{\lambda_{k+1}} (A(r)x - A(t)x) dr \\ &= \left(\frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} \right) \frac{1}{t - \lambda_k} \int_{\lambda_k}^t (A(r)x - A(t)x) dr \\ &\quad + \left(\frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} \right) \frac{1}{\lambda_{k+1} - t} \int_t^{\lambda_{k+1}} (A(r)x - A(t)x) dr. \end{aligned}$$

It follows that $A_\Lambda^S(t)x - A(t)x \rightarrow 0$ as $|\Lambda| \rightarrow 0$. Since almost all points of $[0, T]$ are Lebesgue points of $A(\cdot)x$ the first assertion follows

For the second assestion let $x \in D$ and let Ω be a measurable subset of $[0, T]$. We set $w = x \otimes 1_\Omega$. Then $\|A_\Lambda^S w - Aw\|_{L^p(0, T; X)}^p = \int_\Omega \|A_\Lambda^S(t)x - A(t)x\|_X^p dt \rightarrow 0$ as $|\Lambda| \rightarrow 0$ by *i*) and Lebesgue's Theorem. From which follows that $\|A_\Lambda^S w - Aw\|_{L^p(0, T; X)} \rightarrow 0$ as $|\Lambda| \rightarrow 0$ for all simple function w and thus for all $w \in L^p(0, T; D)$. Let now $w_\Lambda \in L^p(0, T; D)$ such that $w_\Lambda \rightarrow w$ in $L^p(0, T; D)$. Then

$$\|A_\Lambda^S w_\Lambda - Aw\|_{L^p(0, T; X)} \leq C\|w_\Lambda - w\|_{L^p(0, T; D)} + \|A_\Lambda^S w - Aw\|_{L^p(0, T; X)}.$$

Thus (ii) holds. \square

Instead of functions that are constant on each subinterval $[\lambda_k, \lambda_{k+1}[$, we consider now those that are linear.

Lemma 2.2. *Let $A : [0, T] \rightarrow \mathcal{L}(D, X)$ be a bounded and strongly measurable function. Then the following statements hold:*

1. *For all $x \in D$ we have $A_\Lambda^L(t)x \rightarrow A(t)x$ t -a.e. on $[0, T]$ as $|\Lambda| \rightarrow 0$.*
2. *$A_\Lambda^L(\cdot)u_\Lambda(\cdot) \rightarrow A(\cdot)u(\cdot)$ in $L^p(0, T; X)$ as $|\Lambda| \rightarrow 0$ if $u_\Lambda \in L^p(0, T; D)$ such that $u_\Lambda \rightarrow u$ in $L^p(0, T; D)$.*

Proof. Let $x \in D$ and let $t \in [0, T]$ be an arbitrary Lebesgue point of $A(\cdot)x$ and $k \in \{0, 1, \dots, n\}$ be such that $t \in [\lambda_k, \lambda_{k+1})$. Then

$$\begin{aligned} A_\Lambda^L(t)x - A(t)x &= \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} (A_k x - A(t)x) + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} (A_{k+1} x - A(t)x) \\ &= I + II \end{aligned}$$

For the first term I we have

$$I = \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} (A_\Lambda^S(t)x - A(t)x)$$

which converges to zero as $|\Lambda| \rightarrow 0$ by Lemma 2.1. Now we show that II converges also to zero as $|\Lambda|$ goes to 0. Indeed, we have

$$\begin{aligned} A_{k+1}x - A(t)x &= \frac{1}{\lambda_{k+2} - \lambda_{k+1}} \int_t^{\lambda_{k+2}} (A(r) - A(t))x dr \\ &\quad - \frac{1}{\lambda_{k+2} - \lambda_{k+1}} \int_t^{\lambda_{k+1}} (A(r) - A(t))x dr \\ (2.2) \quad &= \left(\frac{\lambda_{k+2} - t}{\lambda_{k+2} - \lambda_{k+1}} \right) \frac{1}{\lambda_{k+2} - t} \int_t^{\lambda_{k+2}} (A(r) - A(t))x dr \\ (2.3) \quad &\quad - \left(\frac{\lambda_{k+1} - t}{\lambda_{k+2} - \lambda_{k+1}} \right) \frac{1}{\lambda_{k+1} - t} \int_t^{\lambda_{k+1}} (A(r) - A(t))x dr \end{aligned}$$

Using again [2, Proposition 1.2.2] we obtain that both terms in (2.2) and (2.3) converges to 0 as $|\Lambda| \rightarrow 0$. Consequently II converges to 0. The claim follows since t is arbitrary Lebesgue point of $A(\cdot)x$. The proof of (2) is the same as the proof of (ii) in Lemma 2.1. \square

3. APPROXIMATION AND CONVERGENCE

In this section H, V are complex separable Hilbert spaces such that $V \hookrightarrow_d H$. Let $T > 0$ and let

$$\mathbf{a} : [0, T] \times V \times V \rightarrow \mathbb{C}$$

be a non-autonomous closed form. This means that $\mathbf{a}(t, \cdot, \cdot)$ is sesquilinear for all $t \in [0, T]$, $\mathbf{a}(\cdot, u, v)$ is measurable for all $u, v \in V$,

$$(3.1) \quad |\mathbf{a}(t, u, v)| \leq M \|u\|_V \|v\|_V \quad (t \in [0, T], u, v \in V)$$

and

$$(3.2) \quad \operatorname{Re} \mathbf{a}(t, u, u) + \omega \|u\| \geq \alpha \|u\|_V^2 \quad (t \in [0, T], u \in V)$$

for some $\alpha > 0, M \geq 0$ and $\omega \in \mathbb{R}$. We assume in addition that \mathbf{a} is *symmetric*; i.e.,

$$\mathbf{a}(t, u, v) = \overline{\mathbf{a}(t, v, u)} \quad (t \in [0, T], u, v \in V).$$

For almost every $t \in [0, T]$ we denote by $\mathcal{A}(t) \in \mathcal{L}(V, V')$ the operator associated with the form $\mathbf{a}(t, \cdot, \cdot)$ in V' . The non-autonomous Cauchy problem (1.2) has *L^2 -maximal regularity in V'* , i.e., for given $f \in L^2(0, T; V')$ and $u_0 \in H$, (1.2) has a unique solution u in $MR(V, V') = L^2(0, T; V) \cap H^1(0, T; V')$. The maximal regularity space $MR(V, V')$ is continuously embedded into $C([0, T], H)$ and if $u \in MR(V, V')$ then the function $\|u(\cdot)\|^2$ is absolutely continuous on $[0, T]$ and

$$(3.3) \quad \frac{d}{dt} \|u(\cdot)\|_H^2 = 2 \operatorname{Re} \langle \dot{u}(\cdot), u(\cdot) \rangle$$

see e.g., [22, Chapter III, Proposition 1.2] or [23, Lemma 5.5.1].

For simplicity we may assume without loss of generality that $\omega = 0$ in (3.2). In fact, let $u \in MR(V, V')$ and let $v := e^{-\omega \cdot} u$. Then $v \in MR(V, V')$ and it satisfies

$$\dot{v}(t) + (\omega + \mathcal{A}(t))v(t) = e^{-\omega t} f(t) \quad t\text{-a.e. on } [0, T], \quad v(0) = 0$$

if and only if u satisfies (1.2). Throughout this section $\omega = 0$ will be our assumption. Let $\Lambda = (0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n+1} = T)$ be a uniform subdivision of $[0, T]$. Let

$$\mathbf{a}_k : V \times V \rightarrow \mathbb{C} \quad \text{for } k = 0, 1, \dots, n$$

be the family of sesquilinear forms given for all $u, v \in V$ and $k = 0, 1, \dots, n$ by

$$(3.4) \quad \mathbf{a}_k(u, v) := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} \mathbf{a}(r; u, v) dr.$$

Remark that \mathbf{a}_k satisfies (3.1) and (3.2) for all $k = 0, 1, \dots, n$. The associated operators are denoted by $\mathcal{A}_k \in \mathcal{L}(V, V')$ and are given for all $u \in V$ and $k = 0, 1, \dots, n$ by

$$(3.5) \quad \mathcal{A}_k u := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} \mathcal{A}(r) u dr.$$

This integral is well defined. Indeed, the mapping $t \mapsto \mathcal{A}(t)$ is strongly measurable by the Pettis Theorem [2] since $t \mapsto \mathcal{A}(t)$ is weakly measurable and the spaces are assumed to be separable. On the other hand, $\|\mathcal{A}(t)u\|_{V'} \leq M\|u\|_V$ for all $u \in V$ and a.e. $t \in [0, T]$. Thus $t \mapsto \mathcal{A}(t)u$ is Bochner integrable on $[0, T]$ with values in V' for all $u \in V$.

The function

$$(3.6) \quad \mathbf{a}_\Lambda^L : [0, T] \times V \times V \rightarrow \mathbb{C}$$

defined for $t \in [\lambda_k, \lambda_{k+1}]$ by

$$(3.7) \quad \mathbf{a}_\Lambda^L(t; u, v) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} \mathbf{a}_k(u, v) + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} \mathbf{a}_{k+1}(u, v) \quad (u, v \in V),$$

is a symmetric non-autonomous closed form and Lipschitz continuous with respect to the time variable $t \in [0, T]$. The associated time dependent operator is denoted by

$$(3.8) \quad \mathcal{A}_\Lambda^L(\cdot) : [0, T] \rightarrow \mathcal{L}(V, V')$$

and is given by

$$(3.9) \quad \mathcal{A}_\Lambda^L(t) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} \mathcal{A}_k + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} \mathcal{A}_{k+1} \quad \text{for } t \in [\lambda_k, \lambda_{k+1}]$$

Since $\mathbf{a}_k, k = 0, 1, \dots, n$ are symmetric, the function $\mathbf{a}_k(v(\cdot))$ belongs to $W^{1,1}(a, b)$ and the following rule formula

$$(3.10) \quad \dot{\mathbf{a}}_k(v(t)) := \frac{d}{dt} \mathbf{a}_k(v(t)) = 2(A_k v(t) | \dot{v}(t)) \quad \text{for a.e. } t \in [a, b],$$

holds whenever $v \in H^1(a, b, H) \cap L^2(a, b, D(A_k))$, for all $[a, b], k = 0, 1, \dots, n$ where A_k is the part of \mathcal{A}_k in H . For the proof we refer to [3, Lemma 3.1].

Theorem 3.1. *Given $f \in L^2(0, T; H)$ and $u_0 \in V$, there is a unique solution $u_\Lambda \in MR(V, H)$ of*

$$(3.11) \quad \dot{u}_\Lambda(t) + \mathcal{A}_\Lambda^L(t) u_\Lambda(t) = f(t), \quad u_\Lambda(0) = u_0.$$

Moreover, $t \mapsto \mathbf{a}_\Lambda(t, u_\Lambda(t)) \in W^{1,2}(0, T)$ and

$$(3.12) \quad 2 \operatorname{Re}(\mathcal{A}_\Lambda^L(t) u_\Lambda(t) | \dot{u}_\Lambda(t))_H = \frac{d}{dt} (\mathbf{a}_\Lambda^L(t; u_\Lambda(t)) - \dot{\mathbf{a}}_\Lambda^L(t; u_\Lambda(t))) \quad t.a.e$$

Proof. The first part of the theorem follows from [18], [5, Theorem 4.2], [15] since $t \mapsto \mathbf{a}_\Lambda^L(t, u, v)$ is piecewise C^1 for all $u, v \in V$. The rule product follows also from [5, Theorem 3.2], but it can be also seen directly from

$$\begin{aligned} \mathbf{a}_\Lambda^L(t; u_\Lambda(t)) &= \int_0^t 2 \operatorname{Re}(\mathcal{A}_\Lambda(s) u_\Lambda(s) | \dot{u}_\Lambda(s))_H ds \\ &\quad + \int_0^t \dot{\mathbf{a}}_\Lambda^L(r, u_\Lambda(r)) dr + \mathbf{a}_\Lambda^L(0, u_0) \quad (t \in [0, T]) \end{aligned}$$

which holds for all $t \in [0, T]$. In fact, let $\delta > 0$, $t \in [0, T]$ be arbitrary and let $l \in \{0, 1, \dots, n\}$ be such that $t \in [\lambda_l, \lambda_{l+1}]$. In order to apply the classical product

rule (3.10), we seek regularizing u_Λ by multiplying with $e^{-\delta A_k}$ and $e^{-\delta A_{k+1}}$. Then

$$\begin{aligned} & \int_{\lambda_k}^{\lambda_{k+1}} (\mathcal{A}_\Lambda(s)u_\Lambda(s)|\dot{u}_\Lambda(s))_H ds \\ &= \lim_{\delta \rightarrow 0} \int_{\lambda_k}^{\lambda_{k+1}} \left(\frac{\lambda_{k+1} - r}{\lambda_{k+1} - \lambda_k} (\mathcal{A}_k e^{-\delta A_k} u_\Lambda(s)|\dot{u}_\Lambda(s))_H \right. \\ & \quad \left. + \frac{r - \lambda_k}{\lambda_{k+1} - \lambda_k} (\mathcal{A}_{k+1} e^{-\delta A_{k+1}} u_\Lambda(s)|\dot{u}_\Lambda(s))_H \right) ds \end{aligned}$$

for $k = 0, 1, \dots, l-1$. Using (3.10) and integrating by part we obtain by an easy calculation

$$\begin{aligned} & 2 \operatorname{Re} \int_{\lambda_k}^{\lambda_{k+1}} (\mathcal{A}_\Lambda(s)u_\Lambda(s)|\dot{u}_\Lambda(s))_H ds \\ &= \lim_{\delta \rightarrow 0} \left[\mathbf{a}_{k+1}(e^{-\frac{\delta}{2} A_{k+1}} u_\Lambda(\lambda_{k+1})) - \mathbf{a}_k(e^{-\frac{\delta}{2} A_k} u_\Lambda(\lambda_k)) \right] \\ & \quad - \lim_{\delta \rightarrow 0} \int_{\lambda_k}^{\lambda_{k+1}} \frac{1}{\lambda_{k+1} - \lambda_k} \left[\mathbf{a}_{k+1}(e^{-\frac{\delta}{2} A_{k+1}} u_\Lambda(s)) - \mathbf{a}_k(e^{-\frac{\delta}{2} A_k} u_\Lambda(s)) \right] ds \\ &= \mathbf{a}_{k+1}(u_\Lambda(\lambda_{k+1})) - \mathbf{a}_k(u_\Lambda(\lambda_k)) - \int_{\lambda_k}^{\lambda_{k+1}} \frac{1}{\lambda_{k+1} - \lambda_k} \left[\mathbf{a}_{k+1}(u_\Lambda(s)) - \mathbf{a}_k(u_\Lambda(s)) \right] ds \\ &= \mathbf{a}_{k+1}(u_\Lambda(\lambda_{k+1})) - \mathbf{a}_k(u_\Lambda(\lambda_k)) - \int_{\lambda_k}^{\lambda_{k+1}} \dot{\mathbf{a}}_\Lambda(s, u_\Lambda(s)) ds \end{aligned}$$

for $k = 0, 2, \dots, l-1$, here we have use that the restriction of $(e^{-tA_k})_{t \geq 0}$ on V is a C_0 -semigroup. By a similar argument as above we obtain for the integral over (λ_l, t)

$$\begin{aligned} & 2 \operatorname{Re} \int_{\lambda_l}^t (\mathcal{A}_\Lambda(s)u_\Lambda(s)|\dot{u}_\Lambda(s))_H ds \\ &= \frac{\lambda_{l+1} - t}{\lambda_{l+1} - \lambda_l} \mathbf{a}_l(u_\Lambda(t)) + \frac{t - \lambda_l}{\lambda_{l+1} - \lambda_l} \mathbf{a}_{l+1}(u_\Lambda(t)) - \mathbf{a}_l(u_\Lambda(\lambda_l)) \\ & \quad - \int_{\lambda_l}^t \frac{1}{\lambda_{l+1} - \lambda_l} \left[\mathbf{a}_{l+1}(u_\Lambda(s)) - \mathbf{a}_l(u_\Lambda(s)) \right] ds \\ &= \mathbf{a}_\Lambda^L(t, u_\Lambda(t)) - \mathbf{a}_l(u_\Lambda(\lambda_l)) - \int_{\lambda_l}^t \dot{\mathbf{a}}_\Lambda(s, u_\Lambda(s)) ds \end{aligned}$$

Consequently

$$\begin{aligned} & 2 \operatorname{Re} \int_0^t (\mathcal{A}_\Lambda(s)u_\Lambda(s)|\dot{u}_\Lambda(s))_H ds \\ &= 2 \operatorname{Re} \sum_{k=0}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} (\mathcal{A}_\Lambda(s)u_\Lambda(s)|\dot{u}_\Lambda(s))_H ds + 2 \operatorname{Re} \int_{\lambda_l}^t (\mathcal{A}_\Lambda(s)u_\Lambda(s)|\dot{u}_\Lambda(s))_H ds \\ &= -\mathbf{a}_0(u_0) + \mathbf{a}_\Lambda^L(t, u_\Lambda(t)) - \int_0^t \dot{\mathbf{a}}_\Lambda(r, u_\Lambda(r)) dr \end{aligned}$$

This completes the proof. \square

The next proposition shows that u_Λ from Theorem 3.1 approximates the solution of (1.2) with respect to the norm of $MR(V, V')$.

Proposition 3.1. *Let $f \in L^2(0, T; H)$ and $u_0 \in V$ and let $u_\Lambda \in MR(V, H)$ be the solution of (3.11). Then u_Λ converges strongly in $MR(V, V')$ as $|\Lambda| \rightarrow 0$ to the solution of (1.2).*

Proof. Let $f \in L^2(0, T; H)$ and $u_0 \in V$. Let $u, u_\Lambda \in MR(V, V')$ be the solution of (1.2) and (3.11) respectively. Set $w_\Lambda := u_\Lambda - u$ and $g_\Lambda := (\mathcal{A} - \mathcal{A}_\Lambda^L)u$. Then $w_\Lambda \in MR(V, V')$ and satisfies

$$\dot{w}_\Lambda(t) + \mathcal{A}_\Lambda^L(t)w_\Lambda(t) = g_\Lambda(t), \quad w_\Lambda(0) = 0.$$

From the product rule (3.3) it follows

$$\begin{aligned} \frac{d}{dt} \|w_\Lambda(t)\|_H^2 &= 2 \operatorname{Re} \langle g_\Lambda(t) - \mathcal{A}_\Lambda^L(t)w_\Lambda(t), w_\Lambda(t) \rangle \\ &= -2 \operatorname{Re} \langle \mathcal{A}_\Lambda^L(t)w_\Lambda(t), w_\Lambda(t) \rangle + 2 \operatorname{Re} \langle g_\Lambda(t), w_\Lambda(t) \rangle \end{aligned}$$

for almost every $t \in [0, T]$. Integrating this equality on $(0, t)$, we obtain

$$\alpha \int_0^t \|w_\Lambda(s)\|_V^2 ds \leq \int_0^t \|g_\Lambda(s)\|_V' \|w_\Lambda(s)\|_V ds.$$

This estimate and the Young's inequality

$$ab \leq \frac{1}{2} \left(\frac{a^2}{\varepsilon} + \varepsilon b^2 \right) \quad (\varepsilon > 0, a, b \in \mathbb{R}).$$

yield the estimate

$$\alpha \|w_\Lambda\|_{L^2(0, T; V)}^2 \leq 1/\alpha \|g_\Lambda\|_{L^2(0, T; V')}^2.$$

The term of the right hand side of this inequality converges by Proposition 2.2 to 0 as $|\Lambda| \rightarrow 0$. It follows that $u_\Lambda \rightarrow u$ strongly in $L^2(0, T; V)$. Again from the second assertion of Proposition 2.2 follows that $\mathcal{A}_\Lambda^L u_\Lambda \rightarrow \mathcal{A}u$ in $L^2(0, T; V')$. Letting $|\Lambda|$ go to 0 in

$$\dot{w}_\Lambda = \dot{u}_\Lambda - \dot{u} = f - \mathcal{A}_\Lambda^L u_\Lambda - \dot{u}$$

and recalling the continuous embedding of $MR(V, V')$ into $C([0, T]; H)$ imply the claim. \square

Next we assume additionally, as in [10] or [11], that there exists a bounded and non-decreasing function $g : [0, T] \rightarrow \mathcal{L}(H)$ such that

$$(3.13) \quad |\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \leq (g(t) - g(s)) \|u\|_V \|v\|_V$$

for $u, v \in V, s, t \in [0, T], s \geq t$. Our aim is to show that under this assumption the solution u_Λ of (3.11) converges weakly in $MR(V, H)$ as $|\Lambda| \rightarrow 0$ and that the limit satisfies (1.2). Without loss of generality, we will assume that $g(0) = 0$. Thus g is positive. Let

$$g_\Lambda^L : [0, T] \rightarrow [0, \infty[$$

denote the analogous function to (3.8) and (3.9) for g . Assume that the subdivision Λ is uniform, i.e., $\lambda_{k+1} - \lambda_k = T/n = |\Lambda|$ for all $k = 0, 1, \dots, n$.

Lemma 3.2.

$$(3.14) \quad |\mathfrak{a}_\Lambda^L(t; u, v) - \mathfrak{a}_\Lambda^L(s; u, v)| \leq [g_\Lambda^L(t) - g_\Lambda^L(s)] \|u\|_V \|v\|_V$$

for all $u, v \in V$ and $t, s \in [0, T]$ with $s \leq t$.

Proof. It suffices to show (3.14) for $t, s \in [\lambda_k, \lambda_{k+1}]$ for some $k \in \{0, 1, \dots, n\}$. The general case where t, s belong to two different subintervals follows immediately. Let $u, v \in V$, then

$$\begin{aligned} \mathbf{a}_\Lambda^L(t; u, v) - \mathbf{a}_\Lambda^L(s; u, v) &= \frac{t-s}{\lambda_{k+1} - \lambda_k} \mathbf{a}_{k+1}(u, v) - \frac{t-s}{\lambda_{k+1} - \lambda_k} \mathbf{a}_k(u, v) \\ &= \frac{t-s}{\lambda_{k+1} - \lambda_k} \frac{n}{T} \int_0^{T/n} [\mathbf{a}(r + \lambda_{k+1}; u, v) - \mathbf{a}(r + \lambda_k; u, v)] dr. \end{aligned}$$

Thus (3.13) implies

$$\begin{aligned} &|\mathbf{a}_\Lambda^L(t; u, v) - \mathbf{a}_\Lambda^L(s; u, v)| \\ &\leq \frac{t-s}{\lambda_{k+1} - \lambda_k} \frac{n}{T} \int_0^{T/n} [g(r + \lambda_{k+1}) - g(r + \lambda_k)] dr \|u\|_V \|v\|_V \\ &= \frac{t-s}{\lambda_{k+1} - \lambda_k} \frac{n}{T} \left[\int_{\lambda_{k+1}}^{\lambda_{k+2}} g(r) dr - \int_{\lambda_k}^{\lambda_{k+1}} g(r) dr \right] \|u\|_V \|v\|_V \\ &= \frac{t-s}{\lambda_{k+1} - \lambda_k} \left[\frac{1}{\lambda_{k+2} - \lambda_{k+1}} \int_{\lambda_{k+1}}^{\lambda_{k+2}} g(r) dr \right. \\ &\quad \left. - \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} g(r) dr \right] \|u\|_V \|v\|_V \\ &= \left[\frac{t-s}{\lambda_{k+1} - \lambda_k} g_{k+1} - \frac{t-s}{\lambda_{k+1} - \lambda_k} g_k \right] \|u\|_V \|v\|_V \\ &= [g_\Lambda^L(t) - g_\Lambda^L(s)] \|u\|_V \|v\|_V \end{aligned}$$

□

The main result of this section is the following

Theorem 3.3. *Assume that the non-autonomous closed form \mathbf{a} is symmetric and satisfies (3.13). Let $f \in L^2(0, T; H)$ and $u_0 \in V$ and let $u_\Lambda \in MR(V, H)$ be the solution of (3.11). Then (u_Λ) converges weakly in $MR(V, H)$ as $|\Lambda| \rightarrow 0$ and $u = \lim_{|\Lambda| \rightarrow 0} u_\Lambda$ satisfies (1.2).*

Proof. a) First since u_Λ satisfies (3.11) then

$$\|\dot{u}_\Lambda(t)\|_H + (A_\Lambda^L(t)u_\Lambda(t) | \dot{u}_\Lambda(t))_H = (f(t) | \dot{u}_\Lambda(t))_H \quad t.a.e$$

The product rule (3.12), Cauchy-Schwartz inequality and Young's inequality imply that for almost every $t \in [0, T]$

$$\|\dot{u}_\Lambda(t)\|_H^2 + \frac{d}{dt}(\mathbf{a}_\Lambda^L(t; u_\Lambda(t))) \leq \|f(t)\|_H^2 + \dot{\mathbf{a}}_\Lambda^L(t; u_\Lambda(t)).$$

Integrating now this inequality on $[0, t]$, it follows that

$$(3.15) \quad \int_0^t \|\dot{u}_\Lambda(r)\|_H^2 dr + \alpha \|u_\Lambda(t)\|_V^2 \leq M \|u_0\|_V^2 + \int_0^t \|f(r)\|_H^2 dr + \int_0^t \dot{\mathbf{a}}_\Lambda^L(r; u_\Lambda(r)) dr$$

where α and M are the constants in (3.1)-(3.2).

b) Note that by construction the derivative $\dot{\mathbf{a}}_\Lambda^L$ of \mathbf{a}_Λ^L equals

$$\dot{\mathbf{a}}_\Lambda^L(r; u) = \frac{\mathbf{a}_{k+1}(u) - \mathbf{a}_k(u)}{\lambda_{k+1} - \lambda_k} \quad \text{for a.e. } r \in [\lambda_k, \lambda_{k+1}], \quad u \in V.$$

Now, let $t \in [0, T]$ be arbitrary and let $l \in \{0, 1, \dots, n\}$ be such that $t \in [\lambda_l, \lambda_{l+1}]$. Then

$$\begin{aligned} \int_0^t \dot{\mathbf{a}}_\Lambda^L(r; u_\Lambda(r)) dr &= \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \dot{\mathbf{a}}_\Lambda^L(r; u_\Lambda(r)) dr + \int_{\lambda_l}^t \dot{\mathbf{a}}_\Lambda^L(r; u_\Lambda(r)) dr \\ &= \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \frac{\mathbf{a}_{k+1}(u_\Lambda(r)) - \mathbf{a}_k(u_\Lambda(r))}{\lambda_{k+1} - \lambda_k} dr \\ &\quad + \int_{\lambda_l}^t \frac{\mathbf{a}_{l+1}(u_\Lambda(r)) - \mathbf{a}_l(u_\Lambda(r))}{\lambda_{l+1} - \lambda_l} dr \\ &= \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \frac{\mathbf{a}_\Lambda^L(\lambda_{k+1}; u_\Lambda(r)) - \mathbf{a}_\Lambda^L(\lambda_k; u_\Lambda(r))}{\lambda_{k+1} - \lambda_k} dr \\ &\quad + \int_{\lambda_l}^t \frac{\mathbf{a}_\Lambda^L(\lambda_{l+1}; u_\Lambda(r)) - \mathbf{a}_\Lambda^L(\lambda_l; u_\Lambda(r))}{\lambda_{l+1} - \lambda_l} dr. \end{aligned}$$

By Lemma 3.2 it follows that

$$\begin{aligned} \int_0^t \dot{\mathbf{a}}_\Lambda^L(r; u_\Lambda(r)) dr &\leq \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \frac{g_\Lambda^L(\lambda_{k+1}) - g_\Lambda^L(\lambda_k)}{\lambda_{k+1} - \lambda_k} \|u_\Lambda(r)\|_V^2 dr \\ &\quad + \int_{\lambda_l}^t \frac{g_\Lambda^L(\lambda_{l+1}) - g_\Lambda^L(\lambda_l)}{\lambda_{l+1} - \lambda_l} \|u_\Lambda(r)\|_V^2 dr \\ &= \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \dot{g}_\Lambda^L(r) \|u_\Lambda(r)\|_V^2 dr + \int_{\lambda_l}^t \dot{g}_\Lambda^L(r) \|u_\Lambda(r)\|_V^2 dr \\ &= \int_0^t \dot{g}_\Lambda^L(r) \|u_\Lambda(r)\|_V^2 dr \end{aligned}$$

c) Using an analogous calculus as in part b) and the fact that

$$\dot{g}_\Lambda^L(r) = \frac{g_{k+1} - g_k}{\lambda_{k+1} - \lambda_k} \quad \text{for a.e. } r \in [\lambda_k, \lambda_{k+1}]$$

we can easily see that

$$(3.16) \quad \int_0^t \dot{g}_\Lambda^L(r) dr \leq g(T)$$

since the function g is positive and non-decreasing.

d) As a consequence of (3.15), the parts b)-c) and Gronwall's lemma it follows that

$$\sup_{t \in [0, T]} \|u_\Lambda(t)\|_V^2 \leq 1/\alpha [M \|u_0\|_V^2 + \int_0^T \|f(r)\|_H^2 dr] \exp(g(T)/\alpha).$$

Inserting this estimate into (3.15), we find that there exists $c = c(\alpha, g(T), M) \geq 0$ such that

$$(3.17) \quad \int_0^T \|\dot{u}_\Lambda(s)\|_H^2 ds \leq c [\|u_0\|_V^2 + \|f\|_{L^2(0, T; H)}^2]$$

Since $u_\Lambda(t) = u_\Lambda(0) + \int_0^t \dot{u}_\Lambda(s) ds$, there exists a constant $c = c(c_H, T)$ with

$$(3.18) \quad \int_0^T \|u_\Lambda(s)\|_H^2 ds \leq c [\|u_0\|_V^2 + \|\dot{u}_\Lambda\|_{L^2(0, T; H)}^2],$$

where c_H is the embedding constant of the embedding of V into H . This estimate and (3.17) yield the estimate

$$(3.19) \quad \|u_\Lambda\|_{H^1(0, T; H)}^2 \leq c [\|u_0\|_V^2 + \|f\|_{L^2(0, T; H)}^2]$$

for some constant $c = c(\alpha, M, c_H, g(T), T) > 0$ independent of the subdivision Λ .
 e) It follows from the parts $a) - d)$ that u_Λ is bounded in $H^1(0, T; H)$. On other hand and as mentioned, Problem (1.2) has a unique solution u in $MR(V, V')$ and we have seen in Proposition 3.1 that $MR(V, H) \ni u_\Lambda \rightarrow u$ in $MR(V, V')$. As a consequence $u \in MR(V, H)$. This completes the proof. \square

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